

# FAIR BILATERAL PRICES IN BERGMAN'S MODEL

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## Abstract

In this paper, we examine the pricing and hedging of a contract in the model proposed by Bergman [1] from the perspective of the hedger and his counterparty with arbitrary initial endowments. We derive inequalities satisfied by unilateral prices of a contract and we give the range for its fair bilateral prices. Our study hinges on results for BSDE driven by a multi-dimensional continuous martingales obtained in [11]. We also derive the pricing PDEs for path-independent contingent claims of European style in a Markovian framework.

**Keywords:** hedging, fair prices, borrowing rate, lending rate, margin agreement, BSDE, PDE

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# 1 Introduction

In Bielecki and Rutkowski [2], the authors introduced a generic nonlinear market model which includes several risky assets, multiple funding accounts and margin accounts (for related studies by other authors, see also [3, 4, 5, 6, 7, 12, 13]). Using a suitable version of the no-arbitrage argument, they first discussed the hedger's fair price for a contract in the market model without collateralization (see Section 3.2 in [2]). Subsequently, for a collateralized contract that can be replicated, they defined the hedger's ex-dividend price (see Section 5 in [2]). It was also shown in [2] that the theory of backward stochastic differential equations (BSDEs) is an important tool to compute the ex-dividend price (see, e.g., Propositions 5.2 and 5.4 in [2]). It is worth mentioning that all the pricing and hedging arguments in [2] are given from the viewpoint of the hedger and no attempt was made there to derive no-arbitrage bounds for unilateral prices.

We consider the problem of pricing and hedging of a derivative contract from the perspective of the hedger and his counterparty. Since we work within a nonlinear trading set-up, where the nonlinearity stems from the different interest rates and collateralization, the hedger's and counterparty's price do not necessarily coincide. Therefore, our goal is to compare the hedger's and counterparty's prices and to derive the range for no-arbitrage prices. As shown by Bergman [1], in the model with different lending and borrowing rates, which is a relatively simple instance of a nonlinear market model, the no-arbitrage price of any contingent claims must belong to an arbitrage band with the upper (resp., lower) bound given by the hedger's (resp., the counterparty's) price of the contract. In a recent paper by Mercurio [9], the author extended some results from [1] by examining the pricing of European options in a model with different lending and borrowing interest rates and under collateralization. As emphasized in related papers [2, 10, 11], in the nonlinear setup, especially in the market with different interest rates and idiosyncratic funding costs for risky assets, the initial endowments of the hedger and the counterparty are important. Unlike in the classic options pricing model, which enjoys linearity, it is no longer sufficient to consider the case of null initial endowments since the ex-dividend prices may depend on initial endowments (see Proposition 5.2 in [2]). Therefore, the results obtained in [1] and [9] are only valid in situation where the initial endowments of the hedger and the counterparty are assumed to be null.

We revisit the market model studied by Bergman [1] and we extend it in several respects. First, we study general collateralized contracts, rather than path-independent European claims. Second, we assume that investors have possibly non-zero (either positive or negative) initial endowments and both parties are allowed to use their initial endowments to invest in risky assets for the purpose of hedging. Finally, we do not assume a priori any particular financial model, but rather we work within an abstract semimartingale set-up. Our main goals are to examine how the initial endowment affects the price and to establish the existence of a non-empty interval for fair bilateral prices. We argue that the properties of their respective prices will be quite different under alternative assumptions about initial endowments of both parties. As in [10], we show that the pricing inequalities can be obtained from the general results for the non-linear BSDEs, which determine unilateral prices and hedging strategies for both parties. For the sake of completeness, we also derive the pricing PDEs for path-independent European claims in a Markovian framework, thus extending once again the approach of Bergman [1].

This work is organized as follows. In Section 2, we introduce our set-up and we recall definitions and results regarding hedging strategies for collateralized contracts in a model with different lending and borrowing rates. We also show there that the model is arbitrage-free for both parties, in the sense of Definition 2.5. For a more extensive discussion of models with funding costs and collateralization, the reader is referred to [2, 10]. In Section 3, we first establish the existence and uniqueness of solutions to BSDEs yielding the ex-dividend prices and hedging strategies for the hedger and the counterparty. Next, we apply the comparison theorem for BSDEs driven by a multi-dimensional martingale established in [11] to derive the range for fair bilateral prices. In Section 4, we place ourselves in a Markovian framework and we postulate that the interest rates are deterministic. Using the non-linear version of the Feynman-Kac formula, we derive the pricing PDEs for both parties and we describe their respective hedging strategies in terms of solutions to these PDEs.

## 2 Trading under Differential Rates and Collateralization

Throughout the paper, we fix a finite trading horizon date  $T > 0$  for our model of the financial market. Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$  models the flow of information available to all traders. For convenience, we assume that the initial  $\sigma$ -field  $\mathcal{G}_0$  is trivial. Moreover, all processes introduced in what follows are implicitly assumed to be  $\mathbb{G}$ -adapted and any semimartingale is assumed to be càdlàg.

**Risky assets.** For  $i = 1, 2, \dots, d$ , we denote by  $S^i$  the *ex-dividend price* of the  $i$ th risky asset with the *cumulative dividend stream*  $A^i$ . The process  $S^i$  is aimed to represent the price of any traded security, such as, stock, stock option, interest rates swap, currency option, cross-currency swap, CDS, CDO, etc.

**Cash accounts.** The riskless *lending* (resp., *borrowing*) *cash account*  $B^l$  (resp.,  $B^b$ ) is used for unsecured lending (resp., borrowing) of cash.

**Assumption 2.1** The price processes of *primary assets* are assumed to satisfy:

- (i) For each  $i = 1, 2, \dots, d$ , the price  $S^i$  is semimartingale and the cumulative dividend stream  $A^i$  is finite variation process with  $A_0^i = 0$ .
- (ii) The riskless accounts  $B^l$  and  $B^b$  are strictly positive and continuous processes of finite variation with  $B_0^l = B_0^b = 1$  for  $i = 1, 2, \dots, d$ .

By a *bilateral financial contract*, or simply a *contract*, we mean an arbitrary càdlàg process  $A$  of finite variation. The process  $A$  is aimed to represent the *cumulative cash flows* of a given contract from time 0 till its maturity date  $T$ . By convention, we set  $A_{0-} = 0$ .

The process  $A$  is assumed to model all cash flows of a given contract, which are either paid out from the wealth or added to the wealth, as seen from the perspective of the *hedger* (recall that the other party is referred to as the *counterparty*). Note that the process  $A$  includes the initial cash flow  $A_0$  of a contract at its inception date  $t_0 = 0$ . For instance, if a contract has the initial *price*  $p$  and stipulates that the hedger will receive cash flows  $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_k$  at times  $t_1, t_2, \dots, t_k \in (0, T]$ , then we set  $A_0 = p$  so that

$$A_t = p + \sum_{l=1}^k \mathbb{1}_{[t_l, T]}(t) \bar{A}_l.$$

The symbol  $p$  is frequently used to emphasize that all future cash flows  $\bar{A}_l$  for  $l = 1, 2, \dots, k$  are explicitly specified by the contract's covenants, but the initial cash flow  $A_0$  is yet to be formally defined and evaluated. Valuation of a contract  $A$  means, in particular, searching for the range of *fair values*  $p$  at time 0 from the viewpoint of either the hedger or the counterparty. Although the valuation paradigm will be the same for the two parties, due either to the asymmetry in their trading costs and opportunities, or the non-linearity of the wealth dynamics, they will typically obtain different sets of fair prices for  $A$ . This is the main objective of our current work.

### 2.1 Collateralization

In this paper, we examine the situation when the hedger and the counterparty enter a contract and either receive or post collateral with the value formally represented by an exogenously given stochastic process  $C$ , which is assumed to be a semimartingale (or, at least, a càdlàg process). The process  $C$  is referred to as either the *margin account* or the *collateral amount*. Let

$$C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^-. \quad (2.1)$$

By convention,  $C_t^+$  is the cash value of collateral received at time  $t$  by the hedger, whereas  $C_t^-$  represents the cash value of collateral posted by him. For simplicity of presentation, it is postulated throughout that only cash collateral may be posted or received (for other conventions, see [2]).

We also make the following natural assumption regarding the state of the margin account at the contract's maturity date.

**Assumption 2.2** The  $\mathbb{G}$ -adapted collateral amount process  $C$  satisfies  $C_T = 0$ .

The equality  $C_T = 0$  ensures that any collateral amount posted is returned in full to its owner at the contract's expiration, provided that the default event does not occur at  $T$ . Of course, if the default event is also modeled, which is not the case in this work, then one needs to specify the closeout payoff as well.

**Remark 2.1** The current financial practice typically requires the collateral amounts to be held in *segregated* margin accounts, so that the hedger, when he is a collateral taker, cannot make use of the collateral amount for trading. Another collateral convention encountered in practice is *rehypothecation*, which refers to the situation where a bank is allowed to reuse the collateral pledged by its counterparties as collateral for its own borrowing. Note that if the hedger is a collateral giver, then a particular convention regarding segregation or rehypothecation is immaterial for the wealth dynamics of his portfolio.

We are in a position to introduce trading strategies based on a finite family of primary assets. For simplicity, all issues are discussed from the perspective of the hedger, unless explicitly stated otherwise. It is clear that to cover the counterparty it suffices to replace  $(A, C)$  by  $(-A, -C)$ . The following definition is a special case of Definition 4.1 in [2]

**Definition 2.1** A *collateralized hedger's trading strategy* is a quadruplet  $(x, \varphi, A, C)$  where a portfolio  $\varphi$ , given by

$$\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l, \psi^{d+1}, \eta^{d+2}) \quad (2.2)$$

is composed of the *risky assets*  $S^i$ ,  $i = 1, 2, \dots, d$ , the *unsecured lending cash account*  $B^l$  the *unsecured borrowing cash account*  $B^b$ , the *collateral accounts*  $B^{c,b}$  and  $B^{c,l}$ , the *borrowing account*  $B^{d+1}$  associated with the posted cash collateral, and the *lending account*  $B^{d+2}$  associated with the received cash collateral.

For a detailed explanation of all terms arising in the definition of a strategy  $\varphi$ , the reader is referred to Section 4.1 in [2]. Let us only mention that if  $B^{c,b} \neq B^{c,l}$ , for example if the hedger post the collateral, he will receives interest from the counterparty determined by  $B^{c,l}$ , that is, the counterparty pays the hedger the interest determined by  $B^{c,l}$  not  $B^{c,b}$ . This creates asymmetric financial environments for the hedger and the counterparty. We make the following standing assumption.

**Assumption 2.3** The accounts  $B^{c,l}$ ,  $B^{c,b}$ ,  $B^{d+1}$  and  $B^{d+2}$  are strictly positive, continuous processes of finite variation with  $B_0^{c,l} = B_0^{c,b} = B_0^{d+1} = B_0^{d+2} = 1$ .

The case of the *cash collateral* is described by the following postulates:

- (i) If the hedger receives at time  $t$  the amount  $C_t^+$  as cash collateral, then he pays to the counterparty interest determined by the amount  $C_t^+$  and the account  $B^{c,b}$ . Under segregation, he receives interest determined by the amount  $C_t^+$  and the account  $B^{d+2}$  and thus  $\eta_t^{d+2} B_t^{d+2} = C_t^+$ . When rehypothecation is considered, the hedger may temporarily (i.e., before the contract's maturity date or the default time, whichever comes first) utilize the cash amount  $C_t^+$  for trading and thus  $\eta^{d+2} = 0$ .
- (ii) If the hedger posts a cash collateral at time  $t$ , then the collateral amount is borrowed from the dedicated collateral borrowing account  $B^{d+1}$ . He receives interest determined by the amount  $C_t^-$  and the collateral account  $B^{c,l}$ . We postulate that

$$\psi_t^{d+1} B_t^{d+1} = -C_t^-. \quad (2.3)$$

## 2.2 Trading Strategies and Wealth Processes

We examine trading from the perspective of the hedger with an arbitrary initial endowment. For the counterparty, we may use similar arguments after replacing  $(A, C)$  by  $(-A, -C)$ .

In the context of a collateralized contract, we find it convenient to introduce:

- (i) the process  $V_t(x, \varphi, A, C)$  representing the hedger's wealth at time  $t$ ,
- (ii) the process  $V_t^P(x, \varphi, A, C)$  representing the value of hedger's portfolio at time  $t$ ,
- (iii) the *adjustment process*  $V_t^C(x, \varphi, A, C) := V_t(x, \varphi, A, C) - V_t^P(x, \varphi, A, C)$ , which is aimed to quantify the impact of the margin account on a trading strategy.

**Definition 2.2** The hedger's *portfolio's value*  $V^P(x, \varphi, A, C)$  is given by

$$V_t^P(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \psi_t^{d+1} B_t^{d+1}. \quad (2.4)$$

The hedger's *wealth*  $V(x, \varphi, A, C)$  equals

$$V_t(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \psi_t^{d+1} B_t^{d+1} + \eta_t^{d+2} B_t^{d+2}. \quad (2.5)$$

In general, the adjustment process  $V^C(x, \varphi, A, C)$  equals

$$V_t^C(x, \varphi, A, C) = \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \eta_t^{d+2} B_t^{d+2} = -C_t + \eta_t^{d+2} B_t^{d+2} \quad (2.6)$$

where  $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$  and  $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ . In what follows, we only consider the case of cash collateral under rehypothecation, that is, we set  $\eta^{d+2} = 0$ . Moreover, for simplicity of presentation, we assume that the collateral borrowing account  $B^{d+1}$  coincides with  $B^b$ , so that we may and do set  $\psi^{d+1} = 0$ .

The self-financing property of the hedger's strategy is defined in terms of the dynamics of the value process. Note that we use here the process  $V^P(x, \varphi, A, C)$ , and not  $V(x, \varphi, A, C)$ , to emphasize the important role of  $V^P(x, \varphi, A, C)$  as the value of the hedger's portfolio of traded assets. Observe also that the equality  $V^P(x, \varphi, A, C) = V(x, \varphi, A, C)$  holds when  $C$  vanishes, that is,  $C = 0$ .

Let the initial endowment of the hedger be denoted by  $x$ . It is now natural to represent a portfolio as  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  with the corresponding wealth process

$$V_t(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l}$$

where  $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$  and  $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$  for all  $t \in [0, T]$ .

**Definition 2.3** The hedger's trading strategy  $(x, \varphi, A, C)$  is *self-financing* whenever the process  $V^P(x, \varphi, A, C)$ , which is given by

$$V_t^P(x, \varphi, A, C) = \sum_{i=1}^d \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b, \quad (2.7)$$

satisfies

$$\begin{aligned} V_t^P(x, \varphi, A, C) &= x + \sum_{i=1}^d \int_0^t \xi_u^i d(S_u^i + A_u^i) + \int_0^t \psi_u^l dB_u^l + \int_0^t \psi_u^b dB_u^b + A_t \\ &\quad + \int_0^t \eta_u^b dB_u^{c,b} + \int_0^t \eta_u^l dB_u^{c,l} - V_t^C(x, \varphi, A, C) \end{aligned}$$

where

$$V_t^C(x, \varphi, A, C) = \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} = -C_t.$$

We make the natural assumption that  $\psi_t^l \geq 0$  and  $\psi_t^b \leq 0$  for all  $t \in [0, T]$ . Since simultaneous lending and borrowing of cash is either formally precluded or it is sub-optimal (if  $r^b \geq r^l$ , as we will postulate in Assumption 2.4), we also postulate that  $\psi_t^l \psi_t^b = 0$  for all  $t \in [0, T]$ . Consequently, using (2.7), we obtain the following equalities

$$\psi_t^l = (B_t^l)^{-1} \left( V_t^p(x, \varphi, A, C) - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \quad \psi_t^b = -(B_t^b)^{-1} \left( V_t^p(x, \varphi, A, C) - \sum_{i=1}^d \xi_t^i S_t^i \right)^-.$$

**Assumption 2.4** The collateral accounts  $B^{c,l}$  and  $B^{c,b}$  satisfy  $B^{c,l} = B^{c,b} = B^c$  where  $B^c$  is absolutely continuous, so that  $dB_t^c = r_t^c B_t^c dt$  for some  $\mathbb{G}$ -adapted process  $r^c$ . The riskless accounts are absolutely continuous, so that they can be represented as  $dB_t^l = r_t^l B_t^l dt$  and  $dB_t^b = r_t^b B_t^b dt$  for some  $\mathbb{G}$ -adapted processes  $r^l$  and  $r^b$  such that  $0 \leq r^l \leq r^b$ .

In view of Assumption 2.4, we have

$$\begin{aligned} F_t^C &:= \int_0^t \eta_u^b dB_u^{c,b} + \int_0^t \eta_u^l dB_u^{c,l} \\ &= - \int_0^t C_u^+ (B_u^{c,b})^{-1} dB_u^{c,b} + \int_0^t C_u^- (B_u^{c,l})^{-1} dB_u^{c,l} \\ &= - \int_0^t C_u (B_u^c)^{-1} dB_u^c = - \int_0^t r_u^c C_u du. \end{aligned} \tag{2.8}$$

For brevity, we will write  $A^C := A + C + F^C$ . Moreover, we introduce the auxiliary processes  $\tilde{S}^{i,l,\text{cld}}$  and  $\tilde{S}^{i,b,\text{cld}}$  for  $i = 1, 2, \dots, d$ , which are given by the following expressions

$$\tilde{S}_t^{i,l,\text{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1} dA_u^i$$

and

$$\tilde{S}_t^{i,b,\text{cld}} := (B_t^b)^{-1} S_t^i + \int_{(0,t]} (B_u^b)^{-1} dA_u^i$$

so that their dynamics are

$$d\tilde{S}_t^{i,l,\text{cld}} = (B_t^l)^{-1} (dS_t^i - r_t^l S_t^i dt + dA_t^i)$$

and

$$d\tilde{S}_t^{i,b,\text{cld}} = (B_t^b)^{-1} (dS_t^i - r_t^b S_t^i dt + dA_t^i).$$

We also denote

$$A_t^{C,l} := \int_{(0,t]} (B_u^l)^{-1} dA_u^C, \quad A_t^{C,b} := \int_{(0,t]} (B_u^b)^{-1} dA_u^C.$$

Under Assumption 2.4, the self-financing condition for the trading strategy  $(x, \varphi, A, C)$  reads

$$\begin{aligned} dV_t^p(x, \varphi, A, C) &= \sum_{i=1}^d \xi_t^i (dS_t^i + dA_t^i) + dA_t^C + r_t^l \left( V_t^p(x, \varphi, A, C) - \sum_{i=1}^d \xi_t^i S_t^i \right)^+ dt \\ &\quad - r_t^b \left( V_t^p(x, \varphi, A, C) - \sum_{i=1}^d \xi_t^i S_t^i \right)^- dt. \end{aligned}$$

This leads to the following proposition whose easy proof is omitted.

**Proposition 2.1** *The process  $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$  satisfies*

$$dY_t^l = \sum_{i=1}^d Z_t^{l,i} d\tilde{S}_t^{i,l,\text{cld}} + G_l(t, Y_t^l, Z_t^l) dt + dA_t^{C,l} \tag{2.9}$$

where  $Z^{l,i} = \xi^i$ ,  $i = 1, 2, \dots, d$  and the mapping  $G_l$  equals, for all  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$ ,

$$G_l(t, y, z) = \sum_{i=1}^d r_t^l (B_t^l)^{-1} z^i S_t^i + (B_t^l)^{-1} \left( r_t^l \left( y B_t^l - \sum_{i=1}^d z^i S_t^i \right)^+ - r_t^b \left( y B_t^l - \sum_{i=1}^d z^i S_t^i \right)^- \right) - r_t^l y.$$

The process  $Y^b := (B^b)^{-1} V^p(x, \varphi, A, C)$  satisfies

$$dY_t^b = \sum_{i=1}^d Z_t^{b,i} d\tilde{S}_t^{i,b,cld} + G_b(t, Y_t^b, Z_t^b) dt + dA_t^{C,b}$$

where  $Z^{b,i} = \xi^i$ ,  $i = 1, 2, \dots, d$  and the mapping  $G_b$  equals, for all  $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$ ,

$$G_b(t, y, z) = \sum_{i=1}^d r_t^b (B_t^b)^{-1} z^i S_t^i + (B_t^b)^{-1} \left( r_t^l \left( y B_t^b - \sum_{i=1}^d z^i S_t^i \right)^+ - r_t^b \left( y B_t^b - \sum_{i=1}^d z^i S_t^i \right)^- \right) - r_t^b y.$$

The concept of the *netted wealth* was introduced in [2] to study the arbitrage-free property of a model.

**Definition 2.4** The *netted wealth*  $V^{\text{net}}(x, \varphi, A, C)$  of a trading strategy  $(x, \varphi, A, C)$  is given by  $V^{\text{net}}(x, \varphi, A, C) := V(x, \varphi, A, C) + V(0, \tilde{\varphi}, -A, -C)$  where  $(0, \tilde{\varphi}, -A, -C)$  is the unique self-financing strategy satisfying the following conditions:

- (i)  $V_0(0, \tilde{\varphi}, -A, -C) = -A_0$ ,
- (ii)  $\tilde{\xi}_t^i = 0$  for all  $i = 1, 2, \dots, d$  and  $t \in [0, T]$ ,
- (iii)  $\tilde{\psi}_t^l \geq 0$ ,  $\tilde{\psi}_t^b \leq 0$  and  $\tilde{\psi}_t^l \tilde{\psi}_t^b = 0$  for all  $t \in [0, T]$ .

It is worth noting that  $V_0^{\text{net}}(x, \varphi, A, C) = x$  for any contract  $(A, C)$  and any strategy  $\varphi$ . The proof of the next lemma is elementary and thus it is omitted (see Lemma 3.1 in [10]).

**Lemma 2.1** We have  $V^{\text{net}}(x, \varphi, A, C) = V(x, \varphi, A, C) + U(A, C)$ , where the  $\mathbb{G}$ -adapted process of finite variation  $U(A, C) = U$  is the unique solution to the following equation

$$U_t = \int_0^t (B_u^l)^{-1} (U_u - C_u)^+ dB_u^l - \int_0^t (B_u^b)^{-1} (U_u - C_u)^- dB_u^b - F_t^C - A_t$$

where  $F^C$  is given by (2.8). Under Assumption 2.4, we obtain

$$U_t = \int_0^t r_u^l (U_u - C_u)^+ du - \int_0^t r_u^b (U_u - C_u)^- du + \int_0^t r_u^c C_u du - A_t.$$

## 2.3 Arbitrage-Free Property

Depending on the signs of the initial endowments, we will formally work under two alternative assumptions regarding a general set-up considered in this work. It is worth noting, however, that these assumptions may in fact be equivalent when a particular model for the dynamics of risky assets is adopted.

**Assumption 2.5** There exists a probability measure  $\tilde{\mathbb{P}}^l$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,l,cld}$ ,  $i = 1, 2, \dots, d$  are  $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -local martingales.

**Assumption 2.6** There exists a probability measure  $\tilde{\mathbb{P}}^b$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,b,cld}$ ,  $i = 1, 2, \dots, d$  are  $(\tilde{\mathbb{P}}^b, \mathbb{G})$ -local martingales.

In the foregoing definition of admissibility, the discounted netted wealth  $\hat{V}^{\text{net}}(x, \varphi, A, C)$  is defined either as  $V^{\text{net}}(x, \varphi, A, C)/B^l$ , if Assumption 2.5 is postulated, or as  $V^{\text{net}}(x, \varphi, A, C)/B^b$ , when Assumption 2.6 is valid. The same notational convention is used in Proposition 2.2.



**Definition 2.5** A self-financing trading strategy  $(x, \varphi, A, C)$  is *admissible for the hedger* whenever the discounted netted wealth process  $\widehat{V}^{net}(x, \varphi, A, C)$  is bounded from below by a constant.

**Definition 2.6** An admissible trading strategy  $(x, \varphi, A, C)$  is an *arbitrage opportunity for the hedger* with respect to  $(A, C)$  whenever

$$\mathbb{P}(V_T^{net}(x, \varphi, A, C) \geq V_T^0(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{net}(x, \varphi, A, C) > V_T^0(x)) > 0$$

where  $V_t^0(x) := x^+ B_t^l - x^- B_t^b$  for all  $t \in [0, T]$ . A market model is said to be *arbitrage-free* for the hedger if there is no arbitrage opportunity for the hedger in regard to any contract  $(A, C)$ .

**Proposition 2.2** We consider the market model introduced in this section under Assumption 2.4.

(i) If Assumption 2.5 holds and  $x_1 \geq 0, x_2 \geq 0$ , then the market model is arbitrage-free with respect to any contract  $(A, C)$  for the hedger and the counterparty.

(ii) If Assumption 2.6 holds and  $x_1 \leq 0, x_2 \geq 0$ , then the market model is arbitrage-free with respect to any contract  $(A, C)$  for the hedger and the counterparty.

*Proof.* We only prove the non-arbitrage property of the model from the perspective of hedger with a positive initial endowment  $x = x_1 \geq 0$ , since all other cases can be proven using analogous arguments. From (2.9) and  $r^l \leq r^b$ , we know that  $V^l(x, \varphi, A, C) = (B^l)^{-1} V^p(x, \varphi, A, C)$  satisfies

$$dV_t^l(x, \varphi, A, C) \leq \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + dA_t^{C,l}.$$

Furthermore, in view of Lemma 2.1, the netted wealth is given by  $V^{net}(x, \varphi, A, C) = V(x, \varphi, A, C) + U(A, C)$ , where in turn the  $\mathbb{G}$ -adapted process of finite variation  $U(A, C)$  is given by Lemma 2.1. Hence the process  $V^{l,net}(x, \varphi, A, C) := (B^l)^{-1} V^{net}(x, \varphi, A, C) = V^l(x, \varphi, A, C) + (B^l)^{-1} U(A, C)$  satisfies

$$dV_t^{l,net}(x, \varphi, A, C) \leq \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} + (r_t^l - r_t^b)(B_t^l)^{-1}(U_t(A, C) - C_t) dt \leq \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,l,\text{cld}}$$

or, more explicitly,

$$(B_t^l)^{-1}(V_t^{net}(x, \varphi, A, C) - x) \leq \sum_{i=1}^d \int_{(0,t]} \xi_t^i d\tilde{S}_t^{i,l,\text{cld}} \quad (2.10)$$

since  $V^{net}(x, \varphi, A, C) = x$ . The assumption that the process  $V^{l,net}$  is bounded from below, implies that the right-hand side in (2.10) is a  $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -supermartingale, which is null at  $t = 0$ . Next, since  $x \geq 0$ , we have that  $V_T^0(x) = B_T^l x$  and thus, from (2.10), we obtain

$$(B_T^l)^{-1}(V_T^{net}(x, \varphi, A, C) - V_T^0(x)) \leq \sum_{i=1}^d \int_{(0,T]} \xi_t^i d\tilde{S}_t^{i,l,\text{cld}}.$$

Since the probability measure  $\tilde{\mathbb{P}}^l$  was assumed to be equivalent to  $\mathbb{P}$ , we conclude that either the equality  $V_T^{net}(x, \varphi, A, C) = V_T^0(x)$  holds or  $\mathbb{P}(V_T^{net}(x, \varphi, A, C) < V_T^0(x)) > 0$ . This means that arbitrage opportunities are precluded and thus the model is arbitrage-free for the hedger in regard to any contract  $(A, C)$ .  $\square$

### 3 Ex-Dividend Prices and Related Pricing BSDEs

The main goal of this section is to show that, under mild technical assumption, the range of fair bilateral prices of a generic collateralized contract  $(A, C)$  is non-empty for some choices of initial endowments of the hedger and the counterparty.



### 3.1 Generic Market Models

To show the existence of a solution to the pricing BSDE, we need to complement Assumptions 2.5 and 2.6 by imposing specific conditions on the underlying market model. In essence, we postulate that the discounted cumulative prices of risky assets are continuous martingales under an equivalent probability measure and its quadratic variation process satisfies suitable technical conditions.

We define the matrix-valued process  $\mathbb{S}$

$$\mathbb{S}_t := \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & S_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_t^d \end{pmatrix}.$$

We will work under the following alternative assumptions regarding the quadratic variation process for continuous martingales  $\tilde{S}^{l,\text{cld}}$  and  $\tilde{S}^{b,\text{cld}}$ . Note that  $*$  stands for the transposition.

**Assumption 3.1** We postulate that:

- (i) the process  $\tilde{S}^{l,\text{cld}}$  is a continuous, square-integrable,  $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -martingale and has the predictable representation property with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^l$ ,
- (ii) there exists an  $\mathbb{R}^{d \times d}$ -valued,  $\mathbb{G}$ -adapted process  $m^l$  such that

$$\langle \tilde{S}^{l,\text{cld}} \rangle_t = \int_0^t m_u^l (m_u^l)^* du \quad (3.1)$$

with the process  $m^l(m^l)^*$  is invertible and satisfies  $m^l(m^l)^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$  where  $\sigma$  is a  $d$ -dimensional square matrix of  $\mathbb{G}$ -adapted processes satisfying the *ellipticity condition*: there exists a constant  $\Lambda > 0$

$$\sum_{i,j=1}^d (\sigma_t \sigma_t^*)_{ij} a_i a_j \geq \Lambda |a|^2 = \Lambda a^* a, \quad \forall a \in \mathbb{R}^d, t \in [0, T]. \quad (3.2)$$

**Assumption 3.2** We postulate that:

- (i) the process  $\tilde{S}^{b,\text{cld}}$  is a continuous, square-integrable  $(\tilde{\mathbb{P}}^b, \mathbb{G})$ -martingale and has the predictable representation property with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^b$ ,
- (ii) there exists an  $\mathbb{R}^{d \times d}$ -valued,  $\mathbb{G}$ -adapted process  $m^b$  such that

$$\langle \tilde{S}^{b,\text{cld}} \rangle_t = \int_0^t m_u^b (m_u^b)^* du \quad (3.3)$$

with the process  $m^b(m^b)^*$  is invertible and satisfies  $m^b(m^b)^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$  where  $\sigma$  is a  $d$ -dimensional square matrix of  $\mathbb{G}$ -adapted processes satisfying the ellipticity condition (3.2).

### 3.2 Prices and Hedging Strategies

Definition of the ex-dividend price for the hedger and the counterparty is based on replication of all cash flows associated with a given contract  $(A, C)$ .

**Definition 3.1** For a fixed  $t \in [0, T]$ , a self-financing trading strategy  $(V_t^0(x) + p_t, \varphi, A - A_t, C)$ , where  $p_t$  is a  $\mathcal{G}_t$ -measurable random variable, is said to *replicate the collateralized contract*  $(A, C)$  on  $[t, T]$  whenever  $V_T(V_t^0(x) + p_t, \varphi, A - A_t, C) = V_T^0(x)$ .

Since we deal here with a non-linear pricing rule, we need to examine separately the pricing problem for each party and take into account their initial endowments. Of course, if we postulate that we work within a linear framework in which all interest rates coincide, that is,  $r^l = r^b = r^c$ , then, as expected, we obtain the equality  $P_t^h(x_1, A, C) = P_t^c(x_1, A, C)$  for every contract  $(A, C)$  and all  $t$ .

**Definition 3.2** Any  $\mathcal{G}_t$ -measurable random variable for which a replicating strategy for  $(A, C)$  over  $[t, T]$  exists is called the *hedger's ex-dividend price* at time  $t$  for a contract  $(A, C)$  and it is denoted by  $P_t^h(x_1, A, C)$ , so that for some  $\varphi$  replicating  $(A, C)$

$$V_T(V_t^0(x_1) + P_t^h(x_1, A, C), \varphi, A - A_t, C) = V_T^0(x_1).$$

For an arbitrary level  $x_2$  of the counterparty's initial endowment and a strategy  $\tilde{\varphi}$  replicating  $(-A, -C)$ , the *counterparty's ex-dividend price*  $P_t^c(x_2, -A, -C)$  at time  $t$  for a contract  $(-A, -C)$  is implicitly given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \tilde{\varphi}, -A + A_t, -C) = V_T^0(x_2).$$

By a *fair bilateral price*, we mean the price level at which no arbitrage opportunity arises for either party. Hence the range of fair bilateral prices at time  $t$  is defined as follows.

**Definition 3.3** The  $\mathcal{G}_t$ -measurable interval

$$\mathcal{R}_t^f(x_1, x_2) := [P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)]$$

is called the *range of fair bilateral prices* at time  $t$  of an OTC contract  $(A, C)$  between the hedger and the counterparty.

We are in a position to state the results furnishing the ex-dividend prices and replicating strategies for the hedger and the counterparty. Their proofs hinge on a combination of results on BSDEs from [11] with arguments used in [2]. It is worth noting that in Propositions 3.1 and 3.2, the pricing BSDE is driven either by the process  $\tilde{S}^{l, \text{cld}}$  or the process  $\tilde{S}^{b, \text{cld}}$ , depending on whether the initial endowment is positive or negative. This is somewhat inconvenient when we wish to compare prices for both parties, and thus we will also derive in Proposition 3.3 pricing BSDEs driven by a common process, denoted by  $\tilde{S}^{\text{cld}}$ . It is fair to acknowledge, however, that the financial interpretation of the auxiliary process  $\tilde{S}^{\text{cld}}$  is not as transparent as that of the discounted cumulative prices  $\tilde{S}^{l, \text{cld}}$  and  $\tilde{S}^{b, \text{cld}}$ , and thus the process should be seen as a purely mathematical artifact.

Following [11], but with  $Q_t = t$ , we denote by  $\hat{\mathcal{H}}_0^{2, d}$  the subspace of all  $\mathbb{R}^d$ -valued,  $\mathbb{G}$ -adapted processes  $X$  with

$$|X|_{\hat{\mathcal{H}}_0^{2, d}}^2 := \mathbb{E}_{\mathbb{P}} \left[ \int_0^T \|X_t\|^2 dt \right] < \infty. \quad (3.4)$$

Also, let  $\hat{L}_0^2$  stand for the space of all real-valued,  $\mathcal{G}_T$ -measurable random variables  $\eta$  such that  $|\eta|_{\hat{L}_0^2}^2 = \mathbb{E}_{\mathbb{P}}(\eta^2) < \infty$ .

**Definition 3.4** A contract  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^l$*  if the process  $A^{C, l}$  belongs to  $\hat{\mathcal{H}}_0^2$  and the random variable  $A_T^{C, l}$  belongs to  $\hat{L}_0^2$  under  $\tilde{\mathbb{P}}^l$ . A contract  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^b$*  if the process  $A^{C, b}$  belongs to  $\hat{\mathcal{H}}_0^2$  and the random variable  $A_T^{C, b}$  belongs to  $\hat{L}_0^2$  under  $\tilde{\mathbb{P}}^b$ .

From now on, we postulate that the processes  $r^l$  and  $r^b$  are nonnegative and bounded.

**Proposition 3.1** (i) *Let the hedger's initial endowment  $x_1 = x \geq 0$  and let Assumption 3.1 be satisfied. Then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^l$ , the hedger's ex-dividend price equals  $P^h(x, A, C) = B^l(Y^{h, l, x} - x) - C$  where  $(Y^{h, l, x}, Z^{h, l, x})$  is the unique solution to the BSDE*

$$\begin{cases} dY_t^{h, l, x} = Z_t^{h, l, x, *} d\tilde{S}_t^{l, \text{cld}} + G_l(t, Y_t^{h, l, x}, Z_t^{h, l, x}) dt + dA_t^{C, l}, \\ Y_T^{h, l, x} = x. \end{cases} \quad (3.5)$$

*The unique replicating strategy for the hedger equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  where for all  $t \in [0, T]$  and  $i = 1, 2, \dots, d$*

$$\xi_t^i = Z_t^{h, l, x, i}, \quad \eta_t^b = -(B_t^{c, b})^{-1} C_t^+, \quad \eta_t^l = (B_t^{c, l})^{-1} C_t^-,$$

and

$$\psi_t^l = (B_t^l)^{-1} \left( B_t^l Y_t^{h,l,x} - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \quad \psi_t^b = -(B_t^b)^{-1} \left( B_t^b Y_t^{h,l,x} - \sum_{i=1}^d \xi_t^i S_t^i \right)^-.$$

(ii) Let the hedger's initial endowment  $x_1 = x \leq 0$  and let Assumption 3.2 be satisfied. Then for any contract  $(A, C)$  admissible under  $\widetilde{\mathbb{P}}^b$ , the hedger's ex-dividend price equals  $P^h(x, A, C) = B^b(Y^{h,b,x} - x) - C$  where  $(Y^{h,b,x}, Z^{h,b,x})$  is the unique solution to the BSDE

$$\begin{cases} dY_t^{h,b,x} = Z_t^{h,b,x,*} d\widetilde{S}_t^{b,cld} + G_b(t, Y_t^{h,b,x}, Z_t^{h,b,x}) dt + dA_t^{C,b}, \\ Y_T^{h,b,x} = x. \end{cases} \quad (3.6)$$

The unique replicating strategy for the hedger equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  where for all  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = Z_t^{h,b,x,i}, \quad \eta_t^b = -(B_t^{c,b})^{-1} C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1} C_t^-,$$

and

$$\psi_t^l = (B_t^l)^{-1} \left( B_t^l Y_t^{h,l,x} - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \quad \psi_t^b = -(B_t^b)^{-1} \left( B_t^b Y_t^{h,l,x} - \sum_{i=1}^d \xi_t^i S_t^i \right)^-.$$

*Proof.* From Theorem 4.1 in [11], we know that if Assumption 3.1 holds, then BSDE (3.5) has a unique solution  $(Y^{h,l,x}, Z^{h,l,x})$ . As in the proof of Proposition 5.2 in [2], we can show that  $P^h(x, A, C) = B^l(Y^{h,l,x} - x)$  and derive the unique replicating strategy  $\varphi$ .  $\square$

**Proposition 3.2** For any value  $x = x_2$  of the initial endowment, the counterparty's ex-dividend price equals

$$P^c(x, -A, -C) = - (B^l(Y^{c,l,x} - x) + C) \mathbf{1}_{\{x \geq 0\}} - (B^b(Y^{c,b,x} - x) + C) \mathbf{1}_{\{x \leq 0\}}$$

where  $(Y^{c,l,x}, Z^{c,l,x})$  and  $(Y^{c,b,x}, Z^{c,b,x})$  are respectively the unique solutions to the BSDEs

$$\begin{cases} dY_t^{c,l,x} = Z_t^{c,l,x,*} d\widetilde{S}_t^{l,cld} + G_l(t, Y_t^{c,l,x}, Z_t^{c,l,x}) dt - dA_t^{C,l}, \\ Y_T^{c,l,x} = x, \end{cases}$$

and

$$\begin{cases} dY_t^{c,b,x} = Z_t^{c,b,x,*} d\widetilde{S}_t^{b,cld} + G_b(t, Y_t^{c,b,x}, Z_t^{c,b,x}) dt - dA_t^{C,b}, \\ Y_T^{c,b,x} = x. \end{cases}$$

The unique replicating strategy for the counterparty equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  where for all  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = Z_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + Z_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}}, \quad \eta_t^b = -(B_t^{c,b})^{-1} C_t^-, \quad \eta_t^l = (B_t^{c,l})^{-1} C_t^+,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( B_t^l Y_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + B_t^b Y_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( B_t^l Y_t^{c,l,x} \mathbf{1}_{\{x \geq 0\}} + B_t^b Y_t^{c,b,x} \mathbf{1}_{\{x \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^-. \end{aligned}$$

*Proof.* The proof of Proposition 3.2 is analogous to the proof of Proposition 3.1 and thus it is omitted.  $\square$

In order to establish the comparison result for ex-dividend prices when the two parties have arbitrary initial endowments, we need a result when the prices are given by solution to two BSDEs driven by the same continuous martingale. To this end, we introduce the following assumption about the underlying financial model.

**Assumption 3.3** We postulate that:

(i) there exists a probability measure  $\tilde{\mathbb{P}}^\beta$  equivalent to  $\mathbb{P}$  such that the processes  $\tilde{S}^{i,\text{cld}}, i = 1, 2, \dots, d$  given by (3.7)

$$d\tilde{S}_t^{i,\text{cld}} = dS_t^i + dA_t^i - \beta_t^i S_t^i dt \quad (3.7)$$

for some  $\mathbb{G}$ -adapted bounded processes  $\beta^i$  satisfying  $r^b \leq \beta^i$ , are  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -continuous square-integrable martingales and have the predictable representation property with respect to the filtration  $\mathbb{G}$  under  $\tilde{\mathbb{P}}^\beta$ ,

(ii) there exists an  $\mathbb{R}^{d \times d}$ -valued,  $\mathbb{G}$ -adapted process  $m$  such that

$$\langle \tilde{S}^{\text{cld}} \rangle_t = \int_0^t m_u m_u^* du \quad (3.8)$$

where  $mm^*$  is invertible and satisfies  $mm^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$  where a  $d$ -dimensional square matrix  $\sigma$  of  $\mathbb{G}$ -adapted processes satisfies the ellipticity condition (3.2).

**Definition 3.5** We say that  $(A, C)$  is *admissible under  $\tilde{\mathbb{P}}^\beta$*  when  $A^C \in \hat{\mathcal{H}}_0^2$  and  $A_T^C \in \hat{L}_0^2$  under  $\tilde{\mathbb{P}}^\beta$ .

The next result expresses the unilateral prices of a contract  $(A, C)$  in terms of solutions to BSDEs driven by the continuous  $\tilde{\mathbb{P}}^\beta$ -martingale  $\tilde{S}^{\text{cld}}$ . It will be used in the next section to study the range of fair bilateral prices. To alleviate notation, we denote

$$G(t, y, z) = r_t^l \left( y - \sum_{i=1}^d z^i S_t^i \right)^+ - r_t^b \left( y - \sum_{i=1}^d z^i S_t^i \right)^-.$$

**Proposition 3.3** *Let Assumption 3.3 be valid. Then for any  $x_1, x_2 \in \mathbb{R}$  and an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ , we have that  $P^h(x_1, A, C) = \tilde{Y}^{h,x_1} - C$  and  $P^c(x_2, -A, -C) = \tilde{Y}^{c,x_2} - C$  where  $(\tilde{Y}^{h,x_1}, \tilde{Z}^{h,x_1})$  is the unique solution to the following BSDE*

$$\begin{cases} d\tilde{Y}_t^{h,x_1} = \tilde{Z}_t^{h,x_1,*} d\tilde{S}_t^{\text{cld}} + G^h(t, x_1, \tilde{Y}_t^{h,x_1}, \tilde{Z}_t^{h,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,x_1} = 0, \end{cases}$$

and  $(\tilde{Y}^{c,x_2}, \tilde{Z}^{c,x_2})$  is the unique solution to the following BSDE

$$\begin{cases} d\tilde{Y}_t^{c,x_2} = \tilde{Z}_t^{c,x_2,*} d\tilde{S}_t^{\text{cld}} + G^c(t, x_2, \tilde{Y}_t^{c,x_2}, \tilde{Z}_t^{c,x_2}) dt + dA_t^C, \\ \tilde{Y}_T^{c,x_2} = 0, \end{cases}$$

where

$$G^h(t, x, y, z) := \sum_{i=1}^d z^i \beta_t^i S_t^i + (-x r_t^l B_t^l + G(t, y + x B_t^l, z)) \mathbb{1}_{\{x \geq 0\}} + (-x r_t^b B_t^b + G(t, y + x B_t^b, z)) \mathbb{1}_{\{x \leq 0\}}$$

and

$$G^c(t, x, y, z) := \sum_{i=1}^d z^i \beta_t^i S_t^i + (x r_t^l B_t^l - G(t, -y + x B_t^l, -z)) \mathbb{1}_{\{x \geq 0\}} + (x r_t^b B_t^b - G(t, -y + x B_t^b, -z)) \mathbb{1}_{\{x \leq 0\}}.$$

The unique replicating strategy for the hedger equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  where for all  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = \tilde{Z}_t^{h,x_1,i}, \quad \eta_t^b = -(B_t^{c,b})^{-1} C_t^+, \quad \eta_t^l = (B_t^{c,l})^{-1} C_t^-,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( \tilde{Y}_t^{h,x_1} + x_1 B_t^l \mathbb{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbb{1}_{\{x_1 \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( \tilde{Y}_t^{h,x_1} + x_1 B_t^l \mathbb{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbb{1}_{\{x_1 \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^-. \end{aligned}$$

The unique replicating strategy for the counterparty equals  $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$  where for all  $t \in [0, T]$  and  $i = 1, 2, \dots, d$

$$\xi_t^i = -\tilde{Z}_t^{c,x_2,i}, \quad \eta_t^b = -(B_t^{c,b})^{-1}C_t^-, \quad \eta_t^l = (B_t^{c,l})^{-1}C_t^+,$$

and

$$\begin{aligned} \psi_t^l &= (B_t^l)^{-1} \left( -\tilde{Y}_t^{c,x_2} + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^+, \\ \psi_t^b &= -(B_t^b)^{-1} \left( -\tilde{Y}_t^{c,x_2} + x_2 B_t^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbf{1}_{\{x_2 \leq 0\}} - \sum_{i=1}^d \xi_t^i S_t^i \right)^-. \end{aligned}$$

### 3.3 Range of Fair Bilateral Prices

We are now in a position to study the range of fair bilateral prices at time  $t$  (see Definition 3.3). It appears that, under suitable assumptions, it is non-empty when the initial endowments of the two parties have the same sign but, in general, it may be empty if the signs are different, that is, when  $x_1 < 0$  and  $x_2 > 0$ .

We first examine the case where the initial endowments satisfy  $x_1 \geq 0$  and  $x_2 \geq 0$ .

**Proposition 3.4** *Let Assumption 3.1 be valid. Then for any  $x_1 \geq 0, x_2 \geq 0$  and an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^l$  we have, for every  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.},$$

so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.

*Proof.* We assume that  $x_1 \geq 0, x_2 \geq 0$  and we denote  $\bar{Y}^{h,l,x_1} := Y^{h,l,x_1} - x_1$  and  $\bar{Z}^{h,l,x_1} = Z^{h,l,x_1}$ . In view of Propositions 3.1 and 3.2, the pair  $(\bar{Y}^{h,l,x_1}, \bar{Z}^{h,l,x_1})$  is the unique solution of the following BSDE

$$\begin{cases} d\bar{Y}_t^{h,l,x_1} = \bar{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{l,\text{cld}} + G_l(t, \bar{Y}_t^{h,l,x_1} + x_1, \bar{Z}_t^{h,l,x_1}) dt + dA_t^{C,l}, \\ \bar{Y}_T^{h,l,x_1} = 0. \end{cases}$$

Similarly,  $(\bar{Y}^{c,l,x_2}, \bar{Z}^{c,l,x_2}) := (-(Y^{c,l,x_2} - x_2), \bar{Z}^{c,l,x_2} = -Z^{c,l,x_2})$  is the unique solution of the following BSDE

$$\begin{cases} d\bar{Y}_t^{c,l,x_2} = \bar{Z}_t^{c,l,x_2,*} d\tilde{S}_t^{l,\text{cld}} - G_l(t, -\bar{Y}_t^{c,l,x_2} + x_2, -\bar{Z}_t^{c,l,x_2}) dt + dA_t^{C,l}, \\ \bar{Y}_T^{c,l,x_2} = 0. \end{cases}$$

In view of the comparison theorem for BSDEs (see Theorem 3.3 in [11]), if we show that  $-G_l(t, y + x_1, z) \geq G_l(t, -y + x_2, -z)$  for all  $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ ,  $\tilde{\mathbb{P}}^l \otimes \ell - \text{a.e.}$ , then we will deduce that  $\bar{Y}^{h,l,x_1} \geq \bar{Y}^{c,l,x_2}$ . We denote

$$\begin{aligned} \delta &:= G_l(t, y + x_1, z) + G_l(t, -y + x_2, -z) \\ &= -r_t^l(x_1 + x_2) + (B_t^l)^{-1}r_t^l(\delta_1^+ + \delta_2^+) - (B_t^l)^{-1}r_t^b(\delta_1^- + \delta_2^-) \end{aligned}$$

where

$$\delta_1 := B_t^l y + B_t^l x_1 - \sum_{i=1}^d z^i S_t^i, \quad \delta_2 := -B_t^l y + B_t^l x_2 + \sum_{i=1}^d z^i S_t^i.$$

Since  $r^l \leq r^b$ , we have

$$\begin{aligned} \delta &= -r_t^l(x_1 + x_2) + (B_t^l)^{-1}r_t^l(\delta_1^+ + \delta_2^+) - (B_t^l)^{-1}r_t^b(\delta_1^- + \delta_2^-) \\ &\leq -r_t^l(x_1 + x_2) + (B_t^l)^{-1}r_t^l(\delta_1 + \delta_2) = 0. \end{aligned}$$

Consequently, we have  $\delta \leq 0$ , which yields  $-G_l(t, y + x_1, z) \geq G_l(t, -y + x_2, -z)$ , the proof is complete.  $\square$

**Proposition 3.5** *Let Assumption 3.2 be valid. Then for any  $x_1 \leq 0, x_2 \leq 0$  and an arbitrary contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^b$  we have, for all  $t \in [0, T]$ ,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^b - \text{a.s.},$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

*Proof.* It is now sufficient to show

$$-G_b(t, y + x_1, z) \geq G_b(t, -y + x_2, -z), \quad \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \tilde{\mathbb{P}}^b \otimes \ell - \text{a.e.}$$

If we denote

$$\delta := G_b(t, y + x_1, z) + G_b(t, -y + x_2, -z),$$

then, using similar arguments as in the proof of Proposition 3.4, we can prove that  $\delta \leq 0$ .  $\square$

Now we consider the case when the initial endowments satisfy  $x_1 \geq 0$  and  $x_2 \leq 0$ .

**Proposition 3.6** *Let Assumption 3.3 hold and the initial endowments satisfy  $x_1 \geq 0, x_2 \leq 0$ . Then the following statements are valid.*

(i) *If  $x_1 x_2 = 0$ , then for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and every  $t \in [0, T]$*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}, \quad (3.9)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

(ii) *Assume that  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . Then inequality (3.9) holds for all contracts  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$  if and only if  $x_1 x_2 = 0$ .*

*Proof.* (i) If  $x_1 \geq 0, x_2 \leq 0$ , then we can show that

$$\begin{aligned} \delta &:= g(t, y + x_1 B_t^l, z) + g(t, -y + x_2 B_t^b, -z) - x_1 r_t^l B_t^l - x_2 r_t^b B_t^b \\ &\leq \min \{ (r_t^l - r_t^b) x_2 B_t^b, (r_t^b - r_t^l) x_1 B_t^l \}. \end{aligned}$$

Indeed, we have

$$\delta = -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b + r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-),$$

where

$$\delta_1 = y + x_1 B_t^l - \sum_{i=1}^d z^i S_t^i, \quad \delta_2 = -y + x_2 B_t^b + \sum_{i=1}^d z^i S_t^i.$$

From  $r^l \leq r^b$ , we obtain

$$\begin{aligned} \delta &:= -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b + r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) \\ &\leq -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b + \min \{ r_t^l (\delta_1 + \delta_2), r_t^b (\delta_1 + \delta_2) \} \\ &= -x_1 r_t^l B_t^l - x_2 r_t^b B_t^b + \min \{ r_t^l (x_1 B_t^l + x_2 B_t^b), r_t^b (x_1 B_t^l + x_2 B_t^b) \} \\ &= \min \{ (r_t^l - r_t^b) x_2 B_t^b, (r_t^b - r_t^l) x_1 B_t^l \}. \end{aligned}$$

If  $x_1 x_2 = 0$ , then the right-hand side of the above inequality is non-positive. Therefore,  $\delta \leq 0$  and thus for any contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$ , from the comparison theorem for BSDEs and Proposition 3.3, we deduce that for every  $t \in [0, T]$

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

(ii) We now assume that the interest rates  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . If  $x_1 x_2 \neq 0$ , then the example examined in the proof of Proposition 5.4 in [10] gives a contract  $(A, C)$ , such that the inequality

$$P_0^c(x_2, -A, -C) > P_0^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

holds in the present framework, so that  $\mathcal{R}_0^p(x_1, x_2)$  is non-empty almost surely.  $\square$

The last result of this subsection deals with the case where  $x_1 \leq 0$  and  $x_2 \geq 0$ .

**Proposition 3.7** *Let Assumption 3.3 be valid and the initial endowments satisfy  $x_1 \leq 0, x_2 \geq 0$ . Then the following statements are valid.*

(i) *If  $x_1 x_2 = 0$ , then for every contract  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}, \quad (3.10)$$

*so that the range of fair bilateral prices  $\mathcal{R}_t^f(x_1, x_2)$  is non-empty almost surely.*

(ii) *Assume that  $r^l$  and  $r^b$  are deterministic and satisfy  $r_t^l < r_t^b$  for all  $t \in [0, T]$ . Then inequality (3.10) holds for all contracts  $(A, C)$  admissible under  $\tilde{\mathbb{P}}^\beta$  and all  $t \in [0, T]$  if and only if  $x_1 x_2 = 0$ .*

## 4 European Claims and Related Pricing PDEs

To alleviate notation, we assume that  $d = 1$ , so that there is only one risky asset  $S = S^1$ . This is not a serious restriction, however, since all results obtained in this subsection can be easily extended to the multi-asset framework. Moreover, we postulate that the interest rates  $r^l$  and  $r^b$  are deterministic and thus the only source of randomness is the Brownian motion appearing in dynamics (4.1) of the risky asset.

For conciseness, we focus here on the valuation and hedging of an uncollateralized European contingent claim, that is, we set  $C = 0$ . A generic path-independent claim of European style pays a single cash flow  $H(S_T)$  on the expiration date  $T > 0$ , so that

$$A_t - A_0 = -H(S_T) \mathbf{1}_{[T, T]}(t).$$

Since we deal here with a Markovian set-up, it is convenient to consider the pricing problem for a contract initiated at a fixed, but otherwise arbitrary, date  $t \in [0, T]$ . For any fixed  $t < T$ , the risky asset  $S$  has the ex-dividend price dynamics under  $\mathbb{P}$  given by the following expression, for  $u \in [t, T]$ ,

$$dS_u = \mu(u, S_u) du + \sigma(u, S_u) dW_u, \quad S_t = s \in \mathcal{O}, \quad (4.1)$$

where  $W$  is a one-dimensional Brownian motion and  $\mathcal{O}$  is the domain of real values that are attainable by the diffusion process  $S$  (usually  $\mathcal{O} = \mathbb{R}_+$ ). Moreover, the coefficients  $\mu$  and  $\sigma$  are such that SDE (4.1) has a unique strong solution. We also assume that the volatility coefficient  $\sigma$  is bounded and bounded away from zero. Finally, the dividend process equals  $A_t^1 = \int_0^t \kappa(u, S_u) du$ .

Our first goal is to derive the hedger's pricing PDE for a path-independent European claim. We observe that

$$d\tilde{S}_u^{\text{cld}} = dS_u + dA_u^1 - \beta(u, S_u) du = (\mu(u, S_u) + \kappa(u, S_u) - \beta(u, S_u)) du + \sigma(u, S_u) dW_u.$$

From the Girsanov theorem, if we denote

$$a_u := (\sigma(u, S_u))^{-1} (\mu(u, S_u) + \kappa(u, S_u) - \beta(u, S_u))$$

and define the probability measure  $\tilde{\mathbb{P}}^\beta$  as

$$\frac{d\tilde{\mathbb{P}}^\beta}{d\mathbb{P}} = \exp \left\{ - \int_t^T a_u dW_u - \frac{1}{2} \int_t^T |a_u|^2 du \right\},$$

then  $\tilde{\mathbb{P}}^\beta$  is equivalent to  $\mathbb{P}$  and the process  $\tilde{W}$  is the Brownian motion under  $\tilde{\mathbb{P}}^\beta$ , where  $d\tilde{W}_u := dW_u + a_u du$ . It is easy to see that

$$d\tilde{S}_u^{\text{cld}} = \sigma(u, S_u) d\tilde{W}_u$$

and thus we conclude that  $\tilde{S}^{\text{cld}}$  is a  $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -martingale and  $\langle \tilde{S}^{\text{cld}} \rangle_u = \int_t^u |\sigma(v, S_v)|^2 dv$ . Therefore, Assumption 3.3 holds, provided that we assume that the Brownian motion  $\tilde{W}$  has the predictable



representation property under  $(\mathbb{G}, \widetilde{\mathbb{P}}^\beta)$ . Of course, the latter assumption is not restrictive in the present setup.

We now consider path-independent claims of European style with the unique cash flow at time  $T$  given as  $H(S_T)$ . From Proposition 3.3, for any  $x_1 \in \mathbb{R}$  we have  $P^h(x_1, A, C) = \widetilde{Y}^{h,x_1}$  where  $(\widetilde{Y}^{h,x_1}, \widetilde{Z}^{h,x_1})$  is the unique solution of following BSDE driven by the Brownian motion  $\widetilde{W}$

$$\begin{cases} d\widetilde{Y}_u^{h,x_1} = \widetilde{Z}_u^{h,x_1} \sigma(u, S_u) d\widetilde{W}_u + G^h(u, x_1, S_u, \widetilde{Y}_u^{h,x_1}, \widetilde{Z}_u^{h,x_1}) du, \\ \widetilde{Y}_T^{h,x_1} = H(S_T), \end{cases} \quad (4.2)$$

where for  $x_1 \geq 0$ ,

$$G^h(u, x_1, s, y, z) := z\beta(u, s) - x_1 r_u^l B_u^l + r_u^l (y + x_1 B_u^l - zs)^+ - r_t^b (y + x_1 B_u^l - zs)^-$$

and for  $x_1 \leq 0$

$$G^h(u, x_1, s, y, z) := z\beta(u, s) - x_1 r_u^b B_u^b + r_u^l (y + x_1 B_u^b - zs)^+ - r_u^b (y + x_1 B_u^b - zs)^-.$$

The unique replicating strategy for the hedger equals  $\varphi = (\xi, \psi^l, \psi^b)$  where for every  $u \in [t, T]$   $\xi_u = \widetilde{Z}_u^{h,x_1}$  and

$$\begin{aligned} \psi_u^l &= (B_u^l)^{-1} \left( \widetilde{Y}_u^{h,x_1} + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} - \xi_u S_u \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( \widetilde{Y}_u^{h,x_1} + x_1 B_u^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbf{1}_{\{x_1 \leq 0\}} - \xi_u S_u \right)^-. \end{aligned}$$

For a fixed  $(t, s) \in [0, T] \times \mathcal{O}$ , the solution  $(\widetilde{Y}^{h,x_1}, \widetilde{Z}^{h,x_1})$  depends on the initial value  $s$  of the stock price at time  $t$ , so that we write  $(\widetilde{Y}^{h,x_1,s}, \widetilde{Z}^{h,x_1,s})$ . If we denote  $(Y_u^{h,x_1,s}, Z_u^{h,x_1,s}) := (\widetilde{Y}_u^{h,x_1,s}, \widetilde{Z}_u^{h,x_1,s} \sigma(u, S_u^{s,t}))$  and

$$\overline{G}^h(u, x_1, s, y, z) = G^h(u, x_1, s, y, z\sigma^{-1}(u, x)),$$

then BSDE (4.2) reduces to

$$\begin{cases} dY_u^{h,x_1,s} = Z_u^{h,x_1,s} d\widetilde{W}_u + \overline{G}^h(u, x_1, S_u^{s,t}, Y_u^{h,x_1,s}, Z_u^{h,x_1,s}) du, \\ Y_T^{h,x_1,s} = H(S_T^{s,t}). \end{cases} \quad (4.3)$$

Using the non-linear Feynman-Kac formula, under suitable smoothness conditions of the coefficients  $\mu, \sigma, \kappa$  and  $\beta$ , we deduce that the hedger's pricing function  $v(t, s) := Y_t^{h,x_1,s}$  belongs to the class  $C^{1,2}([0, T] \times \mathcal{O})$  and solves the following pricing PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \mathcal{L}v(t, s) = \overline{G}^h(t, x_1, s, v(t, s), \sigma(t, s) \frac{\partial v}{\partial s}(t, s)), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}, \end{cases}$$

where

$$\mathcal{L} := \frac{1}{2} \sigma^2(t, s) \frac{\partial^2}{\partial s^2} + (\beta - \kappa)(t, s) \frac{\partial}{\partial s}.$$

Equivalently, the function  $v(t, s)$  satisfies

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2} \sigma^2(t, s) \frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s) \frac{\partial v}{\partial s}(t, s) - x_1 r_t^l B_t^l \mathbf{1}_{\{x_1 \geq 0\}} - x_1 r_t^b B_t^b \mathbf{1}_{\{x_1 \leq 0\}} \\ \quad + r_t^l \left( v(t, s) + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} - s \frac{\partial v}{\partial s}(t, s) \right)^+ \\ \quad - r_t^b \left( v(t, s) + x_1 B_t^l \mathbf{1}_{\{x_1 \geq 0\}} + x_1 B_t^b \mathbf{1}_{\{x_1 \leq 0\}} - s \frac{\partial v}{\partial s}(t, s) \right)^-, & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases} \quad (4.4)$$

Conversely, if a function  $v \in C^{1,2}([0, T] \times \mathcal{O})$  solves PDE (4.4), then  $(v(u, S_u), \sigma(u, S_u) \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (4.3) on  $u \in [t, T]$  where we write  $S = S^{s,t}$ . Therefore,  $(v(u, S_u), \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (4.2). Consequently, the unique replicating strategy for the hedger equals  $\varphi = (\xi, \psi^l, \psi^b)$  where for  $u \in [t, T]$

$$\begin{aligned}\xi_u &= \frac{\partial v}{\partial s}(u, S_u), \\ \psi_u^l &= (B_u^l)^{-1} \left( v(u, S_u) + x_1 B_u^l \mathbb{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbb{1}_{\{x_1 \leq 0\}} - S_u \frac{\partial v}{\partial s}(u, S_u) \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( v(u, S_u) + x_1 B_u^l \mathbb{1}_{\{x_1 \geq 0\}} + x_1 B_u^b \mathbb{1}_{\{x_1 \leq 0\}} - S_u \frac{\partial v}{\partial s}(u, S_u) \right)^-.\end{aligned}\quad (4.5)$$

Let us now consider the pricing problem for the counterparty with an initial endowment  $x_2$ . We now have  $P^c(x_2, -A, -C) = \tilde{Y}^{c,x_2}$ , where  $(\tilde{Y}^{c,x_2}, \tilde{Z}^{c,x_2})$  is the unique solution of following BSDE

$$\begin{cases} d\tilde{Y}_u^{c,x_2} = \tilde{Z}_u^{c,x_2} \sigma(u, S_u) d\tilde{W}_u + G^c(u, x_2, S_u, \tilde{Y}_u^{c,x_2}, \tilde{Z}_u^{c,x_2}) du, \\ \tilde{Y}_T^{c,x_2} = H(S_T), \end{cases}\quad (4.6)$$

where for  $x_2 \geq 0$ ,

$$G^c(u, x_2, s, y, z) := z\beta(u, s) + x_2 r_u^l B_u^l - r_u^l \left( -y + x_2 B_u^l + zs \right)^+ + r_u^b \left( -y + x_2 B_u^l + zs \right)^-$$

and for  $x_2 \leq 0$

$$G^c(u, x_2, s, y, z) := z\beta(u, s) + x_2 r_u^b B_u^b - r_u^l \left( -y + x_2 B_u^b + zs \right)^+ + r_u^b \left( -y + x_2 B_u^b + zs \right)^-.$$

The unique replicating strategy for the counterparty equals  $\varphi = (\xi, \psi^l, \psi^b)$  where, for every  $u \in [t, T]$ ,  $\xi_u = -\tilde{Z}_u^{c,x_2}$  and

$$\begin{aligned}\psi_u^l &= (B_u^l)^{-1} \left( -\tilde{Y}_u^{h,x_2} + x_2 B_u^l \mathbb{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbb{1}_{\{x_2 \leq 0\}} - \xi_u S_u \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( -\tilde{Y}_u^{h,x_2} + x_2 B_u^l \mathbb{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbb{1}_{\{x_2 \leq 0\}} - \xi_u S_u \right)^-.\end{aligned}$$

For a fixed  $(t, s) \in [0, T] \times \mathcal{O}$ , we denote  $(Y_u^{c,x_2,s}, Z_u^{c,x_2,s}) := (\tilde{Y}_u^{c,x_2}, \tilde{Z}_u^{c,x_2} \sigma(u, S_u^{s,t}))$  and

$$\overline{G}^c(u, x_2, s, y, z) = G^c(u, x_2, s, y, z\sigma^{-1}(u, s)).$$

Then BSDE (4.2) reduces to

$$\begin{cases} dY_u^{c,x_2,s} = Z_u^{c,x_2,s} d\tilde{W}_u + \overline{G}^c(u, x_2, S_u^{s,t}, Y_u^{c,x_2,s}, Z_u^{c,x_2,s}) du, \\ Y_T^{c,x_2,s} = H(S_T^{s,t}). \end{cases}\quad (4.7)$$

Under suitable smoothness conditions imposed on the coefficients  $\mu$  and  $\sigma$ , from the Feynman-Kac formula, we deduce that the function  $v(t, s) := Y_t^{c,x_2,s}$  belongs to  $C^{1,2}([0, T] \times \mathcal{O})$  and solves the following PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \mathcal{L}v(t, s) = \overline{G}^c(t, x_2, s, v(t, s), \sigma(t, s) \frac{\partial v}{\partial s}(t, s)), & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases}\quad (4.8)$$

More explicitly,

$$\begin{cases} \frac{\partial v}{\partial t}(t, s) + \frac{1}{2} \sigma^2(t, s) \frac{\partial^2 v}{\partial s^2}(t, s) = \kappa(t, s) \frac{\partial v}{\partial s}(t, s) + x_2 r_t^l B_t^l \mathbb{1}_{\{x_2 \geq 0\}} + x_2 r_t^b B_t^b \mathbb{1}_{\{x_2 \leq 0\}} \\ \quad - r_t^l \left( -v(t, s) + x_2 B_t^l \mathbb{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbb{1}_{\{x_2 \leq 0\}} + s \frac{\partial v}{\partial s}(t, s) \right)^+ \\ \quad + r_t^b \left( -v(t, s) + x_2 B_t^l \mathbb{1}_{\{x_2 \geq 0\}} + x_2 B_t^b \mathbb{1}_{\{x_2 \leq 0\}} + s \frac{\partial v}{\partial s}(t, s) \right)^-, & (t, s) \in [0, T] \times \mathcal{O}, \\ v(T, s) = H(s), & s \in \mathcal{O}. \end{cases}\quad (4.9)$$

Conversely, if a function  $v \in C^{1,2}([0, T] \times \mathcal{O})$  solves PDE (4.9), then  $(v(u, S_u), \sigma(u, S_u) \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (4.7) on  $u \in [t, T]$  where  $S = S^{s,t}$ . Hence  $(v(u, S_u), \frac{\partial v}{\partial s}(u, S_u))$  solves BSDE (4.6) and the unique replicating strategy for the counterparty equals  $\varphi = (\xi, \psi^l, \psi^b)$  where, for every  $u \in [t, T]$ ,

$$\begin{aligned} \xi_u &= -\frac{\partial v}{\partial s}(u, S_u), \\ \psi_u^l &= (B_u^l)^{-1} \left( -v(u, S_u) + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + S_u \frac{\partial v}{\partial s}(u, S_u) \right)^+, \\ \psi_u^b &= -(B_u^b)^{-1} \left( -v(u, S_u) + x_2 B_u^l \mathbf{1}_{\{x_2 \geq 0\}} + x_2 B_u^b \mathbf{1}_{\{x_2 \leq 0\}} + S_u \frac{\partial v}{\partial s}(u, S_u) \right)^-. \end{aligned} \quad (4.10)$$

The following proposition summarizes the above considerations. When  $\kappa = 0$  (that is, the stock pays no dividends) and  $x_1 = x_2 = 0$ , then PDE (4.4) reduces to PDE (5) in Bergman [1]. Therefore, Proposition 4.1 can be seen as a generalization of Proposition 2 in [1].

**Proposition 4.1** *If  $v(t, s) \in C^{1,2}([0, T] \times \mathcal{O})$  is the solution of quasi-linear PDE (4.4), then the hedger's ex-dividend price of the European claim  $H(S_T)$  equals  $v(t, S_t)$  and the unique replicating strategy  $\varphi = (\xi, \psi^l, \psi^b)$  for the hedger is given by (4.5). Similarly, if  $v(t, s) \in C^{1,2}([0, T] \times \mathcal{O})$  is the solution of quasi-linear PDE (4.9), then the counterparty's ex-dividend price of the European claim  $H(S_T)$  equals  $v(t, S_t)$  and the unique replicating strategy  $\varphi = (\xi, \psi^l, \psi^b)$  for the counterparty is given by (4.10).*

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